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New reconstruction formulas for oversampled processes and functions

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Abstract

This paper addresses the reconstruction of band-limited oversampled stationary processes and functions. The reconstruction is performed from a multiperiodic subset of the periodic sampling sequence and from some isolated samples. Reconstruction performance can be characterized at the omitted sample points. The omission of some sample points provides a time-varying nature to the reconstruction formulas. This particular sampling scheme associated to specific interpolation functions result in an exact reconstruction with an arbitrarily tunable convergence rate. Moreover, the convergence properties hold when the reconstruction is performed in the neighbourhood of any lost sample. Indeed, the formulas can be fitted to any sample loss or deterioration by a simple time index translation. Specific expressions of the general reconstruction formula are derived for different process bandwidth ranges.

Keywords: Multiperiodic sampling; Linear interpolation; Oversampling; Stationary processes; Band-limited functions

1. Introduction

The theory of sampling and interpolation of functions or processes was initiated before the 19th century [1]. Developments of functions in terms of their values at integer points have been studied in various mathematical frameworks. For instance, Cauchy [2] has provided such developments in the framework of complex variable theory. This paper deals with the reconstruction of continuous-time band-limited stationary processes and functions from part of their periodically sampled observations. The theory and examples are mainly presented for random processes. However, since the random case implies more complex mathematical developments, the transposition to deterministic functions is straightforward. Simulations have been performed in both cases.

In what follows, $\mathbf{Z} = \{Z(t), t \in \mathbb{R}\}$ is a real or complex stationary zero-mean process with power spectral density $s(\omega)$ defined by

$$E[Z(t)Z^*(t-\tau)] = \int_{-\pi+a}^{\pi-a} e^{i\omega\tau} s(\omega)\, d\omega, \quad 0 < a < \pi. \tag{1}$$

$E[..]$ stands for mathematical expectation and * for complex conjugate. Parameter a is related to $\mathbf{Z}$ oversampling: the bandwidth of $\mathbf{Z}$ is equal to or even smaller than $2(\pi - a)$. For the sake of simplicity and without any loss of generality, the process is assumed to be sampled at a unit rate. The

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linear reconstruction of $Z(t)$ from the observed sample sequence $\{Z(n), n \in \mathbb{Z}\}$ can be obtained by

$$Z(t) = \sum_{n\in\mathbb{Z}} \frac{\sin \pi(t-n)}{\pi(t-n)} Z(n), \quad t \in \mathbb{R}, \tag{2}$$

with a zero mean square error. Eq. (2), referred to as the classical 'sampling formula', is attributed to several scientists (particularly to Shannon in the 1950s and to Lloyd for stationary processes [3]). The reconstruction formula (2) yields a linear reconstruction in the form of an infinite sum of interpolation functions weighted by sample values. In the oversampled case, other formulas with better convergence can be established using different interpolation functions such as raised cosine functions [4]. Besides classical uniform sampling, many sampling schemes have been considered in the literature (see for instance [5–7] and references therein). Particularly, multiperiodic sampling has been studied as partitions of the sampling sequence in a finite number of periodic subsequences [8–11].

This paper proposes new formulas with an arbitrarily tunable convergence rate in the case of oversampled band-limited stationary processes or functions. The proposed sampling scheme can be related to multiperiodic sampling: the reconstruction use a multiperiodic sample subsequence combined with particular isolated samples. The proposed reconstruction formulas are of the form of (2) but some samples are omitted. Consequently, the formulas can be easily fitted to any sample loss or deterioration by an appropriate time index translation. The reconstruction accuracy can be characterized at the omitted sample points. The choice of specific interpolation functions, involving trigonometric and polynomial functions, result in an exact reconstruction even in the neighbourhood of any lost sample. A general reconstruction formula expression is proposed. Specific expressions are derived for given process bandwidth ranges. A given process spectral occupancy leads to a set of reconstruction formulas characterized by the same periodic sample subsequences and increasing polynomial orders. The convergence rate improves when the polynomial order increases.

General formulas are proposed in Section 2. Section 3 gives particular expressions for given spectral occupancies of the observed process and reconstruction for small time values. The formulas can be easily adapted to different reconstruction time values. Section 4 studies the mean square convergence of the new reconstruction formulas and illustrates the new formula behaviour when a sample is lost or deteriorated. The mathematical proofs given in appendices are essentially based on complex integration theory.

2. New reconstruction formulas

Before going through the general expression of the proposed reconstruction formulas, some subsequences of the sample time set as well as some specific interpolation functions have to be defined.

2.1. Multiperiodic sample subsequence

For a unit sample rate, the sample time set is the set of relative integers $\mathbb{Z}$. The reconstruction formulas proposed in this paper use some subsequences of this sample time set. For a given positive integer Q, let define the following sample time subsequence:

$$J^Q_{kl} = Q^k\mathbb{Z} + (lQ^{k-1} - 1), \quad k \in \mathbb{N}^*, \ l = 1, 2, \ldots, Q-1. \tag{3}$$

In this paper, $Q = 2$ for simplicity but reconstruction formulas can be easily derived for other values of Q. Let J_k denote a subsequence obtained for $Q = 2$:

$$J_k = 2^k\mathbb{Z} + (2^{k-1} - 1), \quad k \in \mathbb{N}^*. \tag{4}$$

Thus, for instance:

- J_1 is the sequence of even integers $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$,
- J_2 is the sequence of integers of the form $4m + 1$, $m \in \mathbb{Z}, \{\ldots, -7, -3, 1, 5, 9, \ldots\}, \ldots$

Note that these subsequences are disjoint but do not realize a partition of $\mathbb{Z}$ as shown in Appendix A:

$$J_k \cap J_l = \emptyset \quad \text{for } k \neq l \quad \text{and} \quad \bigcup_{k=1}^{\infty} J_k = \mathbb{Z} - \{-1\}. \tag{5}$$

In what follows, elements of J_k will be denoted by t_{mk}:

$$t_{mk} = 2^k m + (2^{k-1} - 1), \quad m \in \mathbb{Z}, \ k \in \mathbb{N}^*. \tag{6}$$

For a given process or function bandwidth, the reconstruction formulas use the N first subsequences $J_1, J_2, \ldots, J_N$ where N denotes a positive integer related to $\mathbf{Z}$ bandwidth through the parameter a by

$$N \geqslant N_{\min} = \inf\left\{ n \in \mathbb{N}^*; \sum_{k=1}^{n} 2^{-k} > 1 - \frac{a}{\pi} \right\}. \tag{7}$$

Appendix B shows that (7) ensures the reconstruction formula convergence.

2.2. Specific interpolation functions

The reconstruction formulas proposed in this paper use specific interpolation functions, with much complicated expressions than the ones used in the classical reconstruction (2). Therefore, this section is dedicated to the definition of these specific interpolation functions. First, consider a complex polynomial function $P(z)$ of degree p:

$$P(z)=\prod_{l=1}^{p}(z-a_l)\quad \text{where } a_l\in\mathbb{Z}-\left\{\bigcup_{k=1}^{N}J_k\right\},$$
$$a_l\neq a_k \text{ for } l\neq k \text{ and } P(z)=1 \text{ for } p=0. \tag{8}$$

Note that the zeroes of $P(z)$ are distinct sample times apart from the selected multiperiodic subsequence defined in the previous section. Then consider the complex function f given by

$$f(z)=P(z)g(z)\quad \text{with}$$
$$g(z)=\prod_{k=1}^{N}\sin[\pi 2^{-k}(z-2^{k-1}+1)]. \tag{9}$$

The zeroes of this function f are distinct integers corresponding to the following sampling times:

$$f(z)=0\quad \text{for } z\in\bigcup_{k=1}^{N}J_k \text{ or } z=a_n,\ n=1,2,\ldots,p. \tag{10}$$

Both functions $f(z)$ and $P(z)$ will be used in the proposed reconstruction formulas in the next section.

2.3. New reconstruction formulas

Based on Cauchy and Yen studies [2,10], reconstruction formulas of $Z(t)$ are proposed, as an alternative to the classical one (2) or improved version in the oversampled case [4]. The proof of the proposed formula convergence is given in Appendix B. The proposed reconstruction general expression is given by

$$Z(t)=\sum_{n=1}^{p}\frac{f(t)Z(a_n)}{(t-a_n)P'(a_n)g(a_n)}+\sum_{k=1}^{N}\frac{2^k}{\pi}\sum_{m\in\mathbb{Z}}\frac{(-1)^m f(t)Z(t_{mk})}{(t-t_{mk})P(t_{mk})h_k(t_{mk})}, \tag{11}$$

where $P(z), f(z), g(z), a_n, p$ have been defined in (8) and (9), t_{mk} is given by (6), $P'(z)$ denotes the first derivative of $P(z)$ and

$$h_k(t)=\prod_{m=1,m\neq k}^{N}\sin[\pi 2^{-m}(t-2^{m-1}+1)]. \tag{12}$$

The convergence rate improvement with respect to the classical sampling formula depends on polynomial P degree: the higher p the faster the convergence as proved in the next section.

3. Reconstruction formulas for given bandwidths

In this section, based on the general expression (11), some particular reconstruction formulas are given, depending on the spectral bandwidth of the random process $\mathbf{Z}$. This spectral bandwidth is defined in (1) by $[-\pi+a,\pi-a]$. Three cases are considered in what follows: $a>\pi/2$ (referred to as "high oversampling"), $(\pi/4)<a\leqslant(\pi/2)$ (case of a "medium oversampling"), and $(\pi/8)<a\leqslant(\pi/4)$ (the "near-Shannon sampling" case).

3.1. High oversampling

The case where the sampling rate is at least twice the Nyquist rate is first considered. For a given process or function bandwidth, the reconstruction formula (11) uses the N first subsequences of time instants $J_1,J_2,\ldots,J_N$, N being defined in (7). When $a>\pi/2$, (7) yields:

$$N_{\min}=\inf\left\{n\in\mathbb{N}^*;\sum_{k=1}^{n}2^{-k}\geqslant\frac{1}{2}>1-\frac{a}{\pi}\right\}=1. \tag{13}$$

Therefore, only one subsequence can be selected to be used in (11), i.e. $J_1=2\mathbb{Z}$. Note that, in this case, the classical formula (2) can be applied to the subsequence indexed by J_1. For a reconstruction window involving small time values, a possible choice for the polynomial P is $P(t)=t-1$. A particular expression of general formula (11) is then:

$$Z(t)=Z(1)\sin\left(\frac{\pi t}{2}\right)+\sum_{m\in\mathbb{Z}}Z(2m)\frac{\sin[\pi((t-2m)/2)]}{\pi((t-2m)/2)}\frac{t-1}{2m-1}. \tag{14}$$

For this formula, the general term of the infinite sum is equivalent to m^{-2}. With a polynomial function $P(t)$ of degree 2, the convergence rate will be in m^{-3}. For example, $P(t)=(t-1)(t+1)$ can be used:

$$Z(t)=\left[Z(1)\frac{t+1}{2}+Z(-1)\frac{t-1}{2}\right]\sin\frac{\pi t}{2}+\sum_{m\in\mathbb{Z}}Z(2m)\frac{\sin\pi((t-2m)/2)}{\pi((t-2m)/2)}\times\frac{(t-1)(t+1)}{(2m-1)(2m+1)}. \tag{15}$$

Using a polynomial function $P(z)$ of degree p, the weighting term of $Z(2m)$ in the infinite sum of (11) is

$$\frac{\sin(\pi((t-2m)/2))}{\pi((t-2m)/2)}\frac{P(t)}{P(2m)}. \tag{16}$$

Therefore, the convergence rate is a function of p and is in $m^{-(p+1)}$.

3.2. Medium oversampling

The case where $(\pi/4)<a<(\pi/2)$ is now considered. In this case, $N_{\min}=2$ and the periodic sampling subsequences may be

$$J_1=2\mathbb{Z},\quad J_2=4\mathbb{Z}+1. \tag{17}$$

Since J_2 includes the sampling time $t_{02}=1$, the polynomial $P(z)$ cannot be the same as in the previous case. The smaller (in absolute value) time sample outside $J_1\cup J_2$ is -1 leading to $P(z)=z+1$. The general formula (11) becomes

$$Z(t)=Z(-1)\sin\left(\frac{\pi t}{2}\right)\sin\left(\frac{\pi(t-1)}{4}\right)+\sum_{m\in\mathbb{Z}}Z(2m)\frac{\sin(\pi((t-2m)/2))}{\pi((t-2m)/2)}\times\frac{t+1}{2m+1}\frac{\sin(\pi/4(t-1))}{\sin(\pi/4(2m-1))}+\sum_{m\in\mathbb{Z}}Z(4m+1)\frac{\sin(\pi((t-4m-1)/4))}{\pi((t-4m-1)/4)}\times\frac{t+1}{4m+2}\sin\left(\frac{\pi}{2}t\right). \tag{18}$$

For a higher convergence rate, a second-order polynomial P has to be considered. The sampling times $-4,-3,-2,0,1,2,4,5$ belong to $J_1\cup J_2$, and thus $-1,3$ are the lowest potential zeros of $P(z)$. For $P(z)=(z+1)(z-3)$, the general formula becomes:

$$Z(t)=\left[Z(-1)\frac{3-t}{4}+Z(3)\frac{-t-1}{4}\right]\sin\left(\frac{\pi t}{2}\right)\times\sin\left(\frac{\pi(t-1)}{4}\right)+\sum_{m\in\mathbb{Z}}Z(2m)\frac{\sin(\pi/2(t-2m))}{\pi/2(t-2m)}\times\frac{(t+1)(t-3)}{(2m+1)(2m-3)}\frac{\sin(\pi/4(t-1))}{\sin(\pi/4(2m-1))}+\sum_{m\in\mathbb{Z}}Z(4m+1)\frac{\sin(\pi/4(t-4m-1))}{\pi/4(t-4m-1)}\times\frac{(t+1)(t-3)}{(4m+2)(4m-2)}\sin\left(\frac{\pi}{2}t\right). \tag{19}$$

3.3. Near-Shannon sampling

In this case, the sample rate is near the Nyquist frequency with $(\pi/8)<a<(\pi/4)$. The same kind of arguments detailed in Sections 3.1 and 3.2 leads to choose the subsequences $J_1=2\mathbb{Z}$, $J_2=4\mathbb{Z}+1$, $J_3=8\mathbb{Z}+3$ ($N_{\min}=3$) and the following polynomial functions:

$$P(z)=z+1\quad\text{or}\quad P(z)=(z+1)(z-7). \tag{20}$$

In the case of the above proposed polynomial of degree 2, note that only the sampling time -9 is not used among the set of integers going from -16 to 14. Consequently, the method leads to a very little loss of data. The adapted interpolation formula is

$$Z(t)=\left[Z(-1)\frac{-7+t}{8}+Z(7)\frac{t+1}{8}\right]\sin\frac{\pi t}{2}\sin\frac{\pi(t-1)}{4}\times\sin\frac{\pi(t-3)}{8}+\sum_{m\in\mathbb{Z}}Z(2m)\frac{\sin\pi/2(t-2m)}{\pi/2(t-2m)}\times\frac{(t+1)(t-7)}{(2m+1)(2m-7)}\frac{\sin\pi/4(t-1)}{\sin\pi/4(2m-1)}\times\frac{\sin\pi/8(t-3)}{\sin\pi/8(2m-3)}+\sum_{m\in\mathbb{Z}}Z(4m+1)\frac{\sin\pi/4(t-4m-1)}{\pi/4(t-4m-1)}\times\frac{(t+1)(t-7)}{(4m+2)(4m-6)}\frac{\sin\pi/8(t-3)}{\sin\pi/8(4m-2)}\sin\frac{\pi}{2}t+\sum_{m\in\mathbb{Z}}-Z(8m+3)\frac{\sin\pi/8(t-8m-3)}{\pi/8(t-8m-3)}\times\frac{(t+1)(t-7)}{(8m+4)/(8m-4)}\sin\frac{\pi}{2}t\sin\frac{\pi}{4}(t-1). \tag{21}$$

In the next section, the new reconstruction formula convergence rate is studied through simulations. The new formulas are implemented on processes with three different bandwidth ranges as well as on functions.

4. Simulations

In practice, the new reconstruction formulas are implemented using a finite number of observed samples. Therefore, the infinite sums have to be replaced by finite sums, leading to a reconstruction error. Obviously, the larger the number of terms in the finite sums, the lower the error. The same remark holds when implementing in practice the classical reconstruction formula (2). Therefore, the first simulations (Figs. 1–6) estimate the proposed reconstruction formulas mean square error, for random processes with decreasing bandwidths. The performance of the classical reconstruction formula are displayed as a reference. Reconstruction on the interval $[-1,1]$ denoted $\widehat{Z}(t)$ is derived from $2n+1$ samples corresponding to the sample time window $W_n=\{-n,\ldots,-1,0,1,\ldots,n\}$. The mean square error normalized with respect to the signal energy over $[-1,1]$defined by

$$\sigma_Z^2=\int_{-1}^{+1}E[|Z(t)-\widehat{Z}(t)|^2]\,\mathrm{d}t \quad \text{with} \quad \int_{-1}^{+1}E[|Z(t)|^2]\,\mathrm{d}t=1$$

is estimated from 100 000 Monte-Carlo runs.

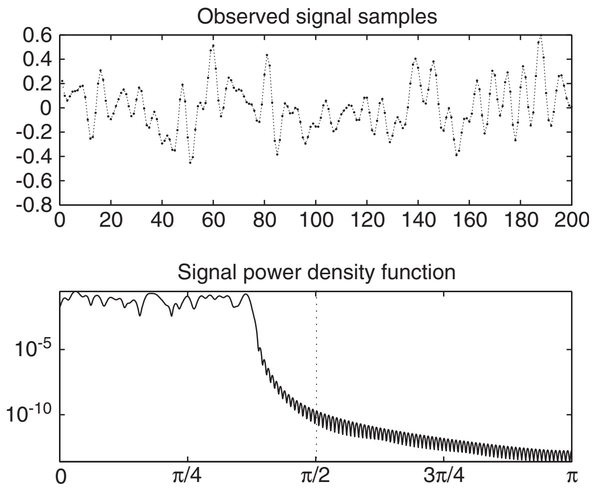

Fig. 1. Filtered white gaussian noise with $a=\pi/2+\pi/8$.

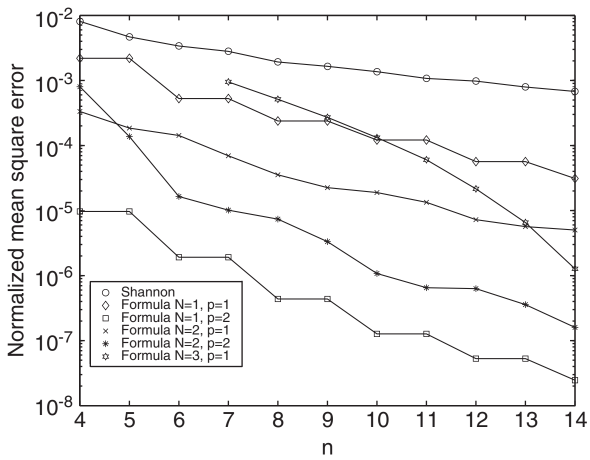

Fig. 2. Normalized mean square error as a function of n.

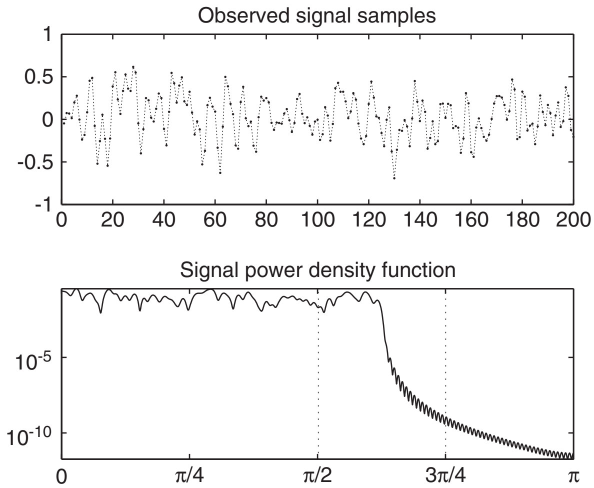

Fig. 3. Filtered white gaussian noise with $a=\pi/8+\pi/16$.

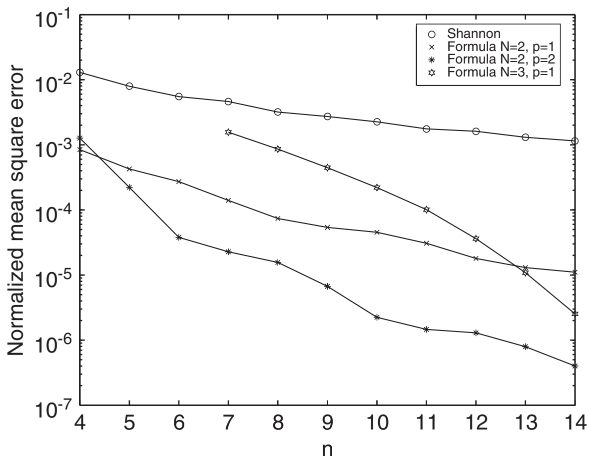

Fig. 4. Normalized mean square error as a function of n.

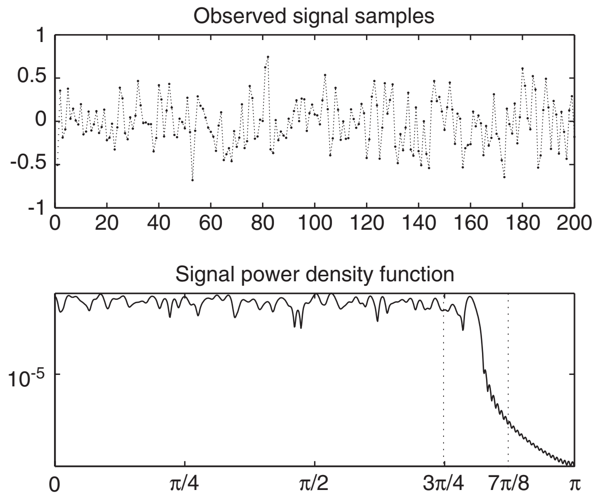

Fig. 5. Filtered white gaussian noise with $a = \pi/8 + \pi/16$.

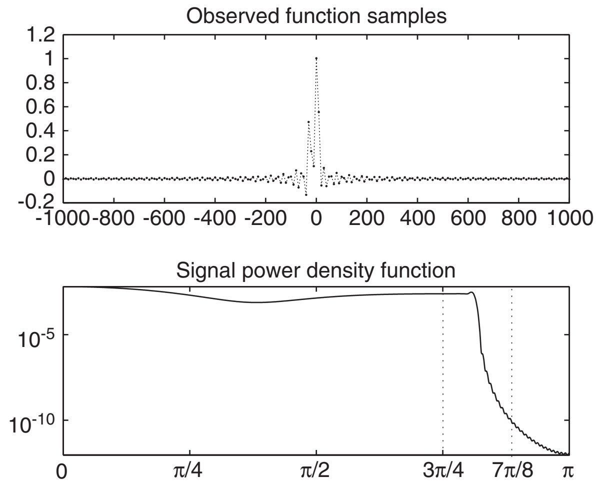

Fig. 7. Near-Shannon sampled function.

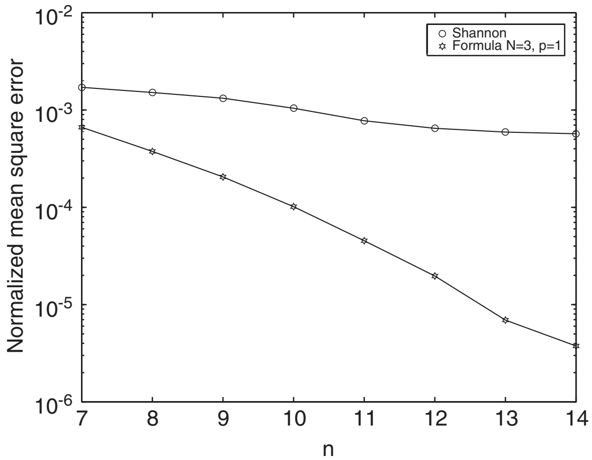

Fig. 6. Normalized mean square error as a function of n.

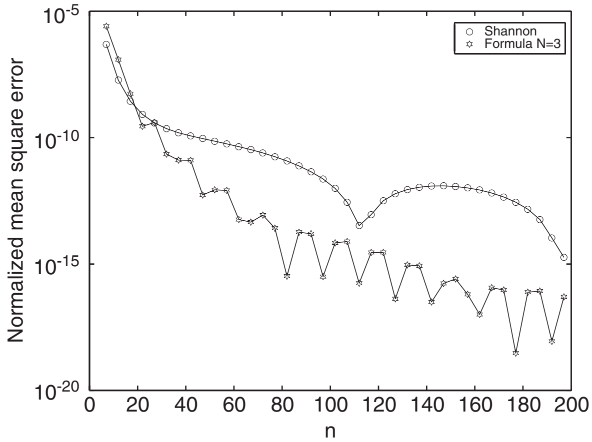

Fig. 8. Normalized mean square error as a function of n.

The simulations are first led with a filtered white noise such that $a = \pi/2 + \pi/8$ (Fig.1). In this case, the performance of the reconstruction formulas with $N = 1, 2, 3$ (given by (14), (15), (18), (19) and (21)) and of the classical sampling formula is estimated for increasing values of n in terms of normalized mean square error (Fig. 2). Note that the formula for $N = 3$ requires $Z(7)$ which imposes a minimum value for n. A filtered white noise such that $a = \pi/4 + \pi/8$ is then considered (Fig. 3). In this case, the performance of the reconstruction formulas for $N = 2, 3$ (given by (18) and (19)) and of the classical sampling formula is estimated (Fig. 4).The case of a near-Shannon sampled process is finally studied, using a filtered white noise such that $a = \pi/8 + \pi/16$ (Fig. 5). In this case, the proposed reconstruction formula (21) is applied.

The simulations displayed on Figs. 7 and 8 estimate Shannon and the new reconstruction formulas performance in the case of a function. A near-Shannon sampled function is considered:

$$f(t) = \sum_{i=1}^{3} \alpha_i \frac{\sin(\Delta\omega(t - t_i))}{\Delta\omega(t - t_i)}, \tag{22}$$

where $\Delta\omega = (\pi - a)/\pi M$ with $a = \pi/8 + \pi/16$ and $M = 10$ is the reconstruction factor (i.e. the reconstruction provides NM samples from N observed samples). The amplitudes and delays of (22) are $\alpha_1 = 1, \alpha_2 = \frac{1}{3}, \alpha_3 = \frac{1}{2}$ and $t_1 = 0, t_2 = 12.5,$

$t_2 = 25$. Note that, in this case, the exact reconstruction error is derived whereas for random processes, the mean square error was estimated through Monte-Carlo runs.

$$\sigma_f^2 = \int_{-1}^{+1} |f(t) - \widehat{f}(t)|^2 \, dt \quad \text{with} \quad \int_{-1}^{+1} |f(t)|^2 \, dt = 1.$$

The convergence rate is clearly improved. However, for very small values of n, Shannon formula leads to a smaller mean square error. Indeed, the proposed formulas asymptotically lead to a better convergence rate. Consequently, the resulting mean square error becomes necessarily smaller from a given length $L_{\min}$ of the observation window (from $L_{\min} = 20$ for the considered functions).

Figs. 2, 4, and 6 clearly show a mean square error improvement with the proposed reconstruction formulas, whatever the observation window length. Moreover, the convergence rate is obviously faster than the convergence rate of the classical formula. For a given value of N, the larger the polynomial order, the better the convergence rate improvement. Note that, in these simulations, the classical formula (2) has been introduced as a reference for performance study. However, this formula derives the convolution between the sample sequence and an appropriate ideal low-pass filter coefficients. Unfortunately, the ideal low-pass filter coefficients decay is very slow resulting in a poor convergence of the reconstruction formula. In the case of oversampled process or functions, the low-pass filter coefficient decay can be optimized. A commonly used reconstruction filter for the oversampled case is the raised cosine filter which provides a quadratic convergence of the reconstruction formula [4]. A raised cosine filter provides a fair performance comparison with the proposed reconstruction formulas in the oversampled case. However, let consider the loss of one sample. As an example, suppose that $Z(15)$ is lost and replaced by zero. Let estimate the reconstruction error in the neighbourhood of this sample, at the time instant $t = 15.5$for instance, i.e.

$$\sigma^2_{Z(15.5)} = E[|Z(15.5) - \widehat{Z}(15.5)|^2].$$

The following numerical results have been obtained from 5000 Monte-Carlo runs.

	CF	RCF	NF(1,1)	NF(1,2)	NF(2,1)	NF(2,2)	NF(3,1)
$\sigma^2_{Z(15.5)}$ no sample loss	0.0660	0.0000	0.0014	0.0011	$0.5656 \times 1.0\text{e} - 3$	$0.3460 \times 1.0\text{e} - 3$	$0.1431 \times 1.0\text{e} - 3$
$\sigma^2_{Z(15.5)}$ $Z(15)$ lost	64.1700	53.7553	0.0014	0.0011	$0.5656 \times 1.0\text{e} - 3$	$0.3460 \times 1.0\text{e} - 3$	$0.1431 \times 1.0\text{e} - 3$

CF denotes the classical formula, RCF the raised cosine filter, and NF(i,j) the new formula for $N = i$, $p = J$. When no sample is lost, the raised cosine filter interpolation outperforms the proposed formulas and thus the classical reconstruction formula. However, the key advantage of the proposed formulas is their adaptation to any sample loss. This simulation demonstrates the influence of a sample loss on the reconstruction accuracy. When a sample is lost, the performance dramatically drop for the time-invariant classical and raised cosine filter reconstruction formulas in a neighbourhood of this sample. The time-varying nature of the proposed reconstruction formulas allows to cope with this situation by an adequate time index translation. Since the proposed formulas use only a part of the available samples, this time translation aims at positioning a useless sample precisely on the lost sample. The consequence is an identical performance with and without this sample loss even in its neighbourhood.

5. Conclusion

This paper addressed the problem of the reconstruction of continuous time processes and functions, from uniformly sampled observations. This problem is studied in the case of band-limited oversampled processes or functions. Within this frame, the classical Shannon interpolation yields an exact reconstruction. However, its convergence rate is low and the reconstruction performance dramatically drops in the neighbourhood of any lost sample. Therefore, this paper proposed some alternative reconstruction formulas with convergence characterized by a tunable rate and insensitivity to a sample loss or deterioration. Depending on the spectral bandwidth of the random processes or of the functions, different reconstruction formulas can be derived. Examples are given in this paper and simulations clearly highlighted the performance of the new reconstruction formulas. However, the key advantage of these new formulas is the performance insensitivity to a sample loss even in the neighbourhood of this sample. This advantage has been illustrated by a performance comparison with the classical and raised cosine filter reconstruction formulas.

Appendix A

This appendix shows particular properties of the subsequences J_n.

Property A1. The subsequences $J_n = 2^n\mathbb{Z} + (2^{n-1} - 1), n \in \mathbb{N}^*$ are disjoint.

Proof. Assume property 1 to be false. In this case, there exist two positive integers m, n $(0<m<n)$ and two relative integers j, k such that

$$2^m j + (2^{m-1} - 1) = 2^n k + (2^{n-1} - 1). \quad (23)$$

This leads to

$$2^{n-m} - 1 = 2[j - 2^{n-m}k], \quad (24)$$

which is impossible, since the first member is odd and the second is even. □

Property A2.

$$\bigcup_{n=1}^{\infty} J_n = \mathbb{Z} - \{-1\}. \quad (25)$$

Proof. In a first step, let m denote an odd positive integer (if m is even, then $m \in J_1$). The dyadic development of m is

$$m = 1 + 2b_1 + 2^2 b_2 + \cdots + 2^{M-2} b_{M-2} + 2^{M-1}, \quad M>0, \quad (26)$$

b_j being 0 or 1.

If $b_j = 1$ for all j, then $m = 2^M - 1 \in J_{M+1}$.

Now, assume that

$$b_1 = \cdots = b_{K-1} = 1, \quad b_K = 0 \text{ and } b_M = 1. \quad (27)$$

with $K<M$, b_j being 0 or 1 for $j = K+1, \ldots, M-1$. Then, obviously $m = 2^K - 1 + 2^{K+1} b_{K+1} + \cdots + 2^M \in J_{K+1}$.

In a second step, let m denote a negative and odd integer, and $m \neq -1$. Since $-m-2$ is positive, there exists K such that $-m-2 \in J_K$ (as proved in the first step). Thus, we can write

$$-m - 2 = j2^K + 2^{K-1} - 1$$
$$\Longleftrightarrow m = (-j-1)2^K + 2^{K-1} - 1. \quad (28)$$

Consequently, $m \in J_K$.

In a third step, let consider the case $m = -1$. If $-1 \in J_K$, then

$$-1 = j2^K + 2^{K-1} - 1 \Leftrightarrow 2^{K-1}(2j+1) = 0$$

which is not possible. □

Appendix B

The sampling formula is closely related to Cauchy works on complex function integration [12]. Let C_n denote the square of the complex plane, which is centred on the origin, with sides parallel to coordinate axes and of length $2(n+\frac{1}{2})$. The following equality (for $t \notin \mathbb{Z}$) can be deduced from the residue theorem:

$$\frac{1}{2i\pi}\int_{C_n} \frac{e^{i\omega z}}{(z-t)f(z)}\,dz = \frac{e^{i\omega t}}{f(t)} + \sum_{k=1}^{p} \operatorname{res}[a_k] + \sum_{k=1}^{N}\sum_{j\in\mathbb{Z}} \operatorname{res}[2^k j + (2^{k-1} - 1)], \quad (29)$$

$$f(z) = \prod_{l=1}^{p}(z - a_l)\prod_{k=1}^{N}\sin[\pi 2^{-k}(z - 2^{k-1} + 1)]. \quad (30)$$

The parameters $\{a_k\}_{k=1,\ldots,p}$ are the (simple) poles of the integrated function due to the polynomial function $P(z)$. Furthermore, $t_{jk} = 2^k j + (2^{k-1} - 1) \in J_k$ is the pole of order 1 due to $\sin \pi 2^{-k}(z - 2^{k-1} + 1)$. The residues can be easily derived. The key point is the definition of the integer N, related to the process or function oversampling through parameter a. Enough subsequences have to be considered to ensure the integral convergence.

On the vertical line defined by

$$\Gamma_n = \{x = 2^{N+n} + \tfrac{1}{2},\ -2^{N+n} - \tfrac{1}{2} \leqslant y \leqslant 2^{N+n} + \tfrac{1}{2}\}, \quad (31)$$

the following inequality holds:

$$\left| \sin \frac{\pi}{2^k}(z - 2^{k-1} + 1) \right| \geqslant \alpha e^{\pi 2^{-k}|y|}, \quad (32)$$

where α is a given positive constant. Hence:

$$\left| \int_{\Gamma_n} \frac{e^{i\omega z}}{(z-t)f(z)} \, dz \right| < \frac{\beta}{|2^{N+n} - |t||} \times \int_{-2^{N+n}-1/2}^{2^{N+n}+1/2} \exp\left[-|y| \left(\varepsilon\omega + \pi \sum_{k=1}^{N} \frac{1}{2^k} \right) \right] dy \underset{n\to\infty}{\longrightarrow} 0, \quad (33)$$

where β is a given positive constant and $\varepsilon = \pm 1$. Indeed, from the hypotheses:

$$\varepsilon\omega + \pi \sum_{k=1}^{N} \frac{1}{2^k} > 0. \quad (34)$$

On the horizontal line defined by:

$$\Gamma'_n = \{-2^{N+n} - \tfrac{1}{2} \leqslant x \leqslant 2^{N+n} + \tfrac{1}{2},\ y = 2^{N+n} + \tfrac{1}{2}\}, \quad (35)$$

a similar result can be demonstrated:

$$\lim_{n\to\infty} \left| \int_{\Gamma'_n} \frac{e^{i\omega z}}{(z-t)f(z)} \, dz \right| = 0. \quad (36)$$

Note that the convergence is uniform for $\omega \in [-\pi + a, \pi - a]$. The general formula (11) is obtained holding (33), (36) in (29) and replacing $e^{i\omega t}$ by $Z(t)$. This last operation is justified using the isometry between the Hilbert space spanned by the process $\mathbf{Z}$ and the L^2-space of functions founded on the power spectrum of $\mathbf{Z}$ (see for example [13] or [14]). Note that the isometry can be used when the spectrum has no density but is contained in $]-\pi + a, \pi - a[$. Of course, the given formulas hold also for functions with the same spectral properties.

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