Power spectra for laser-extinction measurements

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Abstract: Recent laser technology provides accurate measures of the dynamics of fluids and embedded particles. For instance, the laser-extinction measurements (LEM) uses a laser beam passing across the fluid and measures the residual laser light intensity at the fluid output. The particle concentration is estimated from this measurement. However, the particle flow is submitted to random time-varying fluctuations. This study thus proposes to model the received intensity by an appropriate random process. This paper first models the particle flow by a queueing process. Second, the measured intensity power spectrum is derived according to this random model. Finally, the simple case of a constant particle velocity is developped. The proposed model allows to generalize results previously obtained in the litterature with simplified models. Moreover, the particle celerity estimate is provided.

OCIS codes: (120.1740) Combustion diagnostics; (070.6020) Signal processing.

References and links

1. R. J. Adrian, C. S. Yao, "Power spectra of fluid velocities measured by laser Doppler velocimetry," Exp. in Fluids, **5**, 17-28, (1987).
2. A. Chen, J. Hao, Z. Zhou, K. He, "Particle concentration measured from light fluctuations," Opt. Lett. **25**, No. 10, 689-691, (2000).
3. A. Chen, J. Hao, Z. Zhou, J. Zu, "Theoretical solutions for particular scintillation monitors," Opt. Commun. **166**, 15-20, (1999).
4. D. Gross, C. M. Harris, Fundamentals of Queueing Theory, Wiley, 1998.
5. N. Johnson, S.Koltz, Discrete distributions, Houghton mifflin Co. 1969.
6. B. Lacaze, "Spectral properties of scattered light fluctuations," Opt. Commun. **232**, 83-90, (2004).
7. K. Lee, Y. Han, W. Lee, J. Chung, C. Lee, "Quantitative measurements of soot particles in a laminar diffusion flame using LII/LIS technique," Meas. Sci. Technol. **16**, 519-528, (2005).
8. E. Lukacs, Characteristic Functions, Griffin, London, 1970.
9. M. Musculus, L. Pickett, "Diagnostic considerations for optical laser-extinction measurements of soot in high-pressure transient combustion environments," Combustion and Flame, **141**, 371-391, (2005).
10. A. Papoulis, Probability, Random Variables, and Stochastic Processes, McGraw-Hill, 1991.

1. Introduction

The dynamics of fluid and embedded particles is currently studied using laser techniques such as the laser-induced incandescence (LII) [7], the laser-induced scattering (LIS) [7], the laser doppler anemometry (LDA) [1] or the laser-extinction measurements (LEM) [9]. The aim is to determine the properties of emitted particles for in situ monitoring of combustion effluents [3].

This paper focusses on the LEM technique. The LEM system is composed of a laser beam crossing the particle flow. In the opposite direction, an optical device measures the light beam

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intensity power for opacity monitors but also its temporal variation for more recent scintillation monitors [3]. The intersection between the laser beam and the particle flow is a piece of cylinder $\mathscr{V}$ with constant cross-section $\mathscr{S}_1$ (Fig. 1).

When the laser beam crosses the fluid, the light is partially absorbed. The power spectrum

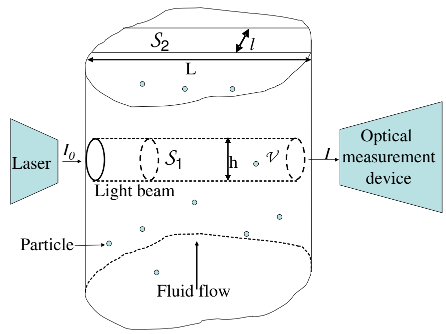

Fig. 1. Laser extinction measurement system

of the residual light intensity I in the emission direction provides an estimate of the particle concentration i.e. of the mean number of particles per volume unit (Fig. 1). From [3] and others, the received intensity I can be written as

$$I = I_0 \prod_{k=1}^{N} (1 - A_k) \qquad A_k = \frac{B_k}{\mathscr{S}_1} < 1 \tag{1}$$

where I_0 denotes the incident laser beam intensity and N is the number of illuminated particles. For a given particle with index k, B_k denotes the extinction cross-section. Uncertainties on the particle shape and orientation result in a random model for B_k. The factor $(1 - A_k)$ models the laser beam attenuation induced by this particle. Let $E[..]$ denote the mathematical expectation. Let define

$$\begin{array}{ll} m_{1B} = E[B_k], & m_{2B} = E\left[B_k^2\right] \\ m_{1A} = \frac{m_{1B}}{\mathscr{S}_1}, & m_{2A} = \frac{m_{2B}}{\mathscr{S}_1^2} \end{array} \tag{2}$$

Then m_{1B} and m_{2B} characterize the particle mechanical properties whereas m_{1A} and m_{2A} characterize the interaction between the light and the particles. Moreover, the relative particle and laser beam motion leads to a random number of illuminated particles N [3]. This study proposes random and as well time-varying models for the number of illuminated particles and the received intensity. Indeed, the received intensity is appropriately modelled by the following random process:

$$I(t) = I_0 \prod_{k \in \mathbf{J}_t} (1 - A_k) \tag{3}$$

where $\mathbf{J}_t$ is the random index set of the illuminated particles at a given time t. The proposed model considers two characteristics of each particle B_k the extinction cross-section and $(1 - A_k)$ the laser beam attenuation induced by this particle as well as a random number of illuminated particles. Note that the resulting optical medium model can be described by the classical scattering and absorption coefficients. These so-called optical characteristics are functions of the proposed model parameters according to appropriate physics considerations.

The following section proposes a random process model for the particle flow. This model leads to the expression of the residual light intensity power spectrum and related quantities as functions of the particle and light beam properties. Particle properties can be estimated from these expressions.

2. The general case

2.1. The queueing process model

Let define t_n and t'_n the input and output times of the n^{th} particle in the illuminated volume $\mathscr{V}$, from a given time origin. Hence:

$$\mathbf{J}_t = \left\{n, t_n < t, t'_n \geq t\right\} \tag{4}$$

Assume that the particle concentration ρ is constant in the whole fluid flow. Under this hypothesis, the input time sequence $\mathbf{t} = \{t_n, n \in \mathbb{Z}\}$ can be modeled as an Homogeneous Poisson Process (HPP) [10] with parameter λ function of ρ and of other physical parameters. The inter-arrival times are exponentially distributed with mean value given by:

$$E\left[t_{n+1} - t_n\right] = \frac{1}{\lambda} \tag{5}$$

The particle flow in the illuminated volume $\mathscr{V}$ can be described by a $M/G/\infty$ queueing process [4], [6]. The particle input times t_n are the "customers arrival times" whereas the particle lightening durations $t'_n - t_n$ are the "service duration" in the queueing denomination. According to queueing notations, M is for an HPP. G denotes the unspecified cumulative distribution of the independent "service durations":

$$G(x) = \Pr\left[t'_n - t_n < x\right] \tag{6}$$

i.e. $G(x)$ characterizes the lightening duration with mean $m' = E\left[t'_n - t_n\right]$. $G(u)$ depends on the particle celerity and on $\mathscr{S}_1$ geometric properties. Finally, ∞ is for an unlimited number of stations i.e. "the service is instantaneous and equivalent for each customer" which means that the particle is not stopped when entering $\mathscr{V}$. Under these assumptions, the number of elements of $\mathbf{J}_t$ i.e. the number of illuminated particles at time t follows a Poisson distribution independently of the distribution G. This last property justifies the results obtained in [2], [3].

Now, assume that the random variables $\{A_n, n \in \mathbb{Z}\}$ are independent identically distributed and are independent of the input times $\{t_n, n \in \mathbb{Z}\}$ and of the service times $\{t'_n - t_n, n \in \mathbb{Z}\}$. The mean value and correlation function of the received intensity $I(t)$ can then be derived leading to the power spectrum. The proofs are given in the appendix. The following section provides these quantities and the relations with the physical parameters.

2.2. The intensity related measurements

2.2.1. The intensity mean value and correlation function

This section first provides the mean m_{1I} and correlation function $K_I(\tau)$ of the measured intensity in the queueing process stationary state:

$$m_{1I} = E\left[I(t)\right] \qquad K_I(\tau) = E\left[I(t)\, I(t+\tau)\right] \tag{7}$$

The appendix proves the main result, for $\tau > 0$:

$$K_I(\tau) = I_0^2 \exp\left[-\lambda\left(m'\left(2m_{1A} - m_{2A}\right) + m_{2A}\int_0^{\tau}\left(1 - G(u)\right)du\right)\right] \tag{8}$$

$K_I(\tau)$ depends on the medium, on the particle and laser beam shapes. [2] and [3] consider the particular cases $\tau = 0$ and ∞. Indeed $K_I(0)$ is the received intensity power whereas

$$\lim_{\tau\to\infty} K_I(\tau) = m_{1I}^2 \tag{9}$$

Then, (8) leads to the intensity first and second order moments:

$$\begin{cases} m_{1I} = I_0 \exp\left[-\lambda m' m_{1A}\right] \\ m_{2I} = K_I(0) = I_0^2 \exp\left[\lambda m' (m_{2A} - 2m_{1A})\right] \end{cases} \tag{10}$$

These relations explain the results obtained with scintillation monitors as shown in the following subsection.

Note that for λ large enough, (8) leads to the following relation:

$$K_I\left(\frac{\tau}{\lambda}\right) \exp\left[-\lambda m' (m_{2A} - 2m_{1A})\right] \simeq I_0^2 \exp\left[-m_{2A}\tau\right] \tag{11}$$

This correlation function leads to a Lorentzian spectrum as shown below.

2.2.2. Laser scintillation measurement

The scintillation measurement have been recently introduced because of its insensitivity to the optical receiver opacity. The system performance is thus independent of the lens contamination caused by dust accumulation. The scintillation is defined by:

$$\Delta = \frac{var[I]}{m_{1I}^2} = \frac{m_{2I} - m_{1I}^2}{m_{1I}^2} \tag{12}$$

where m_{1I}^2 and m_{2I} have been derived in the previous subsection 10, leading to:

$$\Delta = \exp\left[\lambda m' m_{2A}\right] - 1 \tag{13}$$

For a weak extinction, i.e when $\lambda m' m_{2A}$ is small with respect to 1, a limited development leads to:

$$\Delta \cong \lambda m' m_{2A} \tag{14}$$

This relation is in perfect agreement with the result in [3]: the scintillation measurement is independent of the incident intensity I_0 and of m_{1A}. The scintillation relates to the particle physical properties through m_{2A} and to the particle celerity and illuminated volume geometry through m'.

An appropriate system calibration and the a priori knowledge of the particle physical properties allow to estimate λ and thus ρ from Δ measurement. Indeed for a random celerity v, the parameter λ (related to the particle input HPP) $G(x)$ and m' (related to the stay inside the laser beam) can generally be derived under the hypothesis that v is independent of the particle dimensions. In this case, the correlation function and the power spectrum of $I(t)$ can be derived as functions of the physical parameters [6]. Section 3 develops the simplified case of the constant celerity model.

The following subsection derives the received intensity power spectrum.

2.2.3. The intensity power spectrum

The power spectrum of $I(t)$ is the Fourier transform of $K_I(\tau)$. Since $m_{1I} > 0$, the power spectrum $s_I(\omega)$ is composed of a mass at the origin point (equal to m_{1I}^2) added to a spectral density such as:

$$s_I(\omega) = m_{1I}^2 \delta(\omega) + \frac{1}{\pi}\int_0^\infty \left[K_I(\tau) - m_{1I}^2\right] \cos\omega\tau d\tau \tag{15}$$

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The power spectrum is deduced from (8) and (15):

$$s_I(\omega) = I_0^2 e^{-2\lambda m' m_{1A}} \left[\delta(\omega) + \frac{1}{\pi} \int_0^\infty \left(e^{\lambda m_{2A} \left(\int_\tau^\infty (1-G(u)) du \right)} - 1 \right) \cos \omega \tau d\tau \right]. \qquad (16)$$

For λ large enough, according to (11), (16) can be approached by the following Lorentzian spectrum:

$$s_I(\omega) \simeq I_0^2 \frac{e^{-\lambda \left(m'(2m_{1A} - m_{2A}) \right)}}{\pi} \frac{\lambda m_{2A}}{\omega^2 + (\lambda m_{2A})^2} \qquad (17)$$

Figure 2 displays $s_I(\omega)$ given in (17) for $I_0 = 1$, $m\prime = 1$, $m_{1A} = m_{2A} = 1$ and different values of λ.

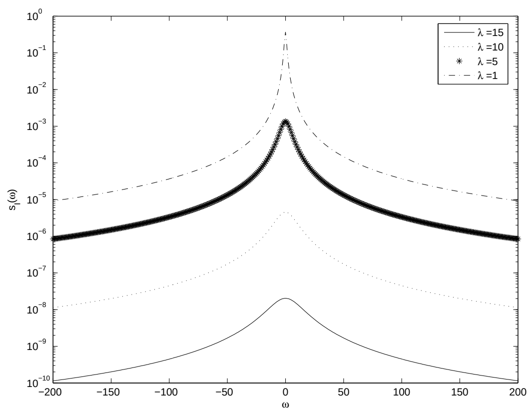

Fig. 2. Approached Lorentzian intensity power spectrum

The following section develops the case of a particle constant celerity.

3. The constant celerity model

3.1. Particle concentration estimation

Consider the illuminated volume $\mathscr{V}$ displayed in Fig. 1. Now, assume that the particles have a constant celerity v perpendicular to the cylinder axis. Let $\mathscr{S}_2$ denote the projection of $\mathscr{V}$ perpendicularly to the celerity direction. Boundary effects are neglected leading to a rectangular approximation for $\mathscr{S}_2$. Let L and l denote $\mathscr{S}_2$ length and width respectively (Fig. 3).

For this simplified model, λ is the mean number of particles in a cylinder of basis $\mathscr{S}_2$ and height v. Consequently, λ is proportional to the particle concentration ρ:

$$\lambda = \rho v l L \qquad (18)$$

Let X denote the particle input abscissa in the laser beam (Fig. 3). X is uniformly distributed on $(0, l)$. Consequently, conditional expectations allow to derive m' the mean fluid crossing duration:

$$\begin{gathered} m' = E\left[E\left[t'_n - t_n \,|X\right]\right] = E\left[\frac{1}{v}\left(f_2(X) - f_1(X)\right)\right] \\ m' = \frac{1}{v}\int_0^l \left(f_2(x) - f_1(x)\right) \frac{dx}{l} = \frac{\mathscr{S}_1}{lv} \end{gathered} \qquad (19)$$

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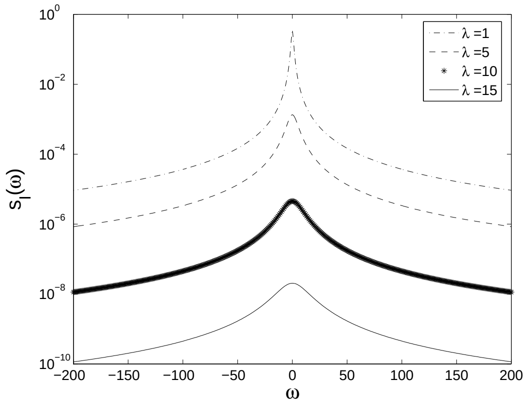

Fig. 3. Constant celerity model notations

Note that m' depends on the cross-section area $\mathscr{S}_1$ and not on its shape. Consequently, in the weak extinction case (14) leads to:

$$\Delta = \frac{m_{2I} - m_{1I}^2}{m_{1I}^2} = \exp\left[\frac{\rho L}{\mathscr{S}_1} m_{2B}\right] - 1 \cong \frac{\rho L}{\mathscr{S}_1} m_{2B}. \tag{20}$$

Therefore, Δ is independent of the celerity v. The measurement of Δ, together with m_{2B} estimation, provide the particle concentration ρ.

3.2. *Particle celerity estimation*

In the case of constant particle celerity, the correlation function expresses as

$$K_I(\tau) = \begin{cases} I_0^2 \exp\left[\rho L \left(-2m_{1B} + \frac{l}{\mathscr{S}_1^2} m_{2B} \alpha(\tau)\right)\right], & 0 < \tau < h/v \\ I_0^2 \exp\left[-2\rho L m_{1B}\right], & \tau > h/v \end{cases} \tag{21}$$

where $\alpha(\tau) = v \int_\tau^{h/v} (1 - G(u))\, du$, h denotes the finite "height" of $\mathscr{S}_1$ (Fig. 1). In the constant celerity case, h/v is the finite maximum fluid crossing duration. Now let consider Δ_τ the generalization of Δ such as:

$$\Delta_\tau = \frac{K_I(\tau) - m_{1I}^2}{m_{1I}^2} = \begin{cases} \exp\left[\frac{lL}{\mathscr{S}_1^2} \rho m_{2B} \alpha(\tau)\right] - 1 & 0 < \tau < h/v \\ 0 & \tau > h/v \end{cases}$$

Like Δ, Δ_τ is independent of the emitted laser intensity I_0. As a function of the product ρm_{2B}, Δ_τ does not provide additionnal information for ρ estimation. Nevertheless, it provides an estimation of the celerity v. Indeed, Δ_τ decreases from Δ to 0 when τ goes from 0 to h/v which provides a simple estimation of the particle celerity. Moreover, the system can be calibrated, for a given value of ρ, which allows to measure m_{2B}.

3.3. *The intensity power spectrum*

(16) leads to the following intensity power spectrum:

$$s_I(\omega) = I_0^2 e^{-2\rho L m_{1B}} \left[\delta(\omega) + \frac{1}{\pi} \int_0^{\frac{h}{v}} \left(\exp\left\{\frac{\rho L l}{\mathscr{S}_1^2} m_{2B} \alpha(\tau)\right\} - 1\right) \cos \omega \tau d\tau\right] \tag{22}$$

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$\alpha(\tau)$ and thus $s_I(\omega)$ depend on the laser beam cross-section shape. As an example, for a unit area rectangle (with side lengths l and $1/l$), the autocorrelation expresses as

$$K_I(\tau) = \begin{cases} I_0^2 \exp\left[\rho L \left\{-2m_{1B} + m_{2B}(1 - lv\tau)\right\}\right] & \tau < 1/lv \\ I_0^2 \exp\left[-2\rho L m_{1B}\right] & \tau > 1/lv \end{cases} \tag{23}$$

Then, $s_I(\omega)$ can be written as

$$s_I(\omega) = I_0^2 e^{-2d}\left[\delta(\omega) + \frac{1}{\pi}\frac{1}{lv}g\left(\frac{\omega}{lv}\right)\right] \text{ with} \begin{cases} g(\omega) = -\frac{\sin\omega}{\omega} + \frac{de^d - d\cos\omega + \omega\sin\omega}{\omega^2 + d^2} \\ d = \rho L m_{2B} \end{cases} \tag{24}$$

The continuous part of the spectrum depends on the product lv, and flattens out when lv increases. In the same time, the discrete part stays unchanged like $E[I(t)]$. Fig. 4 displays the continuous part of the spectrum for, $L = 1$, $l = 1$, $\rho = 0.7$, $m_{2B} = 1$ and different celerity values.

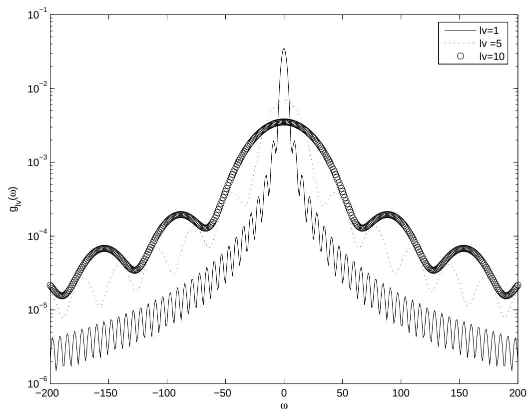

Fig. 4. Continuous part of the intensity power spectrum - Constant celerity case

4. Conclusion

Laser extinction measurements allow to evaluate the volumic concentration of particles in a fluid. In this paper, particles in the laser beam are modelled by customers in a queueing process with an infinite number of stations and Poissonian system inputs. Papers [2] and [3] proposed extinction measurements based on limit values of the intensity autocorrelation function related to the intensity mean value and variance. The result was an estimation of the particle number concentration. The proposed paper provides the expression of the autocorrelation function for each time lag. This allows to derive the intensity power spectrum as a function of the model parameters for each frequency. Then other parameters (cross-section and celerity statistics, particle properties... depending on the problem at hand) can be deduced from the intensity power spectrum measurement obtained from a photo-receiver followed by a spectrum analyzer. For instance, estimates of the particle concentration and celerity have been proposed as well as the received intensity power spectra in the general and constant celerity cases. This paper is theoretical as an extension of papers [2] and [3]. It provides a general framework that can be applied to many different measurement problems. It could be of advantage to check these theoretical results using experimental measurements with different optics and/or particles for specific applications. A similar model has been used for LII [6], and could be extended to more complex systems.

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5. Appendix

Let $|\mathbf{E}|$ denote the cardinal number of the set **E,** and $\overline{\mathbf{E}}$ its complementary set. Now, let define the sets **B, C, D,** by

$$\left\{\begin{array}{l} \mathbf{J}_t = \{n, t_n < t, t'_n \geq t\} \\ \mathbf{B} = \mathbf{J}_t \cap \mathbf{J}_{t+\tau} = \{n, t_n < t, t'_n \geq t+\tau\} \\ \mathbf{C} = \mathbf{J}_t \cap \overline{\mathbf{J}}_{t+\tau} = \{n, t_n < t, t \leq t'_n < t+\tau\} \\ \mathbf{D} = \overline{\mathbf{J}}_t \cap \mathbf{J}_{t+\tau} = \{n, t \leq t_n < t+\tau, t'_n \geq t+\tau\} \\ N(t,\tau) = |\{n, t \leq t_n < t+\tau\}| \end{array}\right. \tag{25}$$

$N(t,\tau)$ is the number of system input times in the interval $[t, t+\tau[$. With these definitions, (3) yields

$$E[I(t)I(t+\tau)] = E\left[\prod_{k\in\mathbf{B}}(1-A_k)^2 \prod_{j\in\mathbf{C}}(1-A_j) \prod_{m\in\mathbf{D}}(1-A_m)\right] \tag{26}$$

A basic property of the HPP is the independence of $N(0,t)$ and $N(t,\tau)$, which implies the independence of the sets $\mathbf{J}_t \cap \mathbf{J}_{t+\tau}$ and $\mathbf{J}_t \cap \overline{\mathbf{J}}_{t+\tau}$ with $\overline{\mathbf{J}}_t \cap \mathbf{J}_{t+\tau}$. Consequently, a conditional expectation leads to

$$E[I(t)I(t+\tau)] = E\left[(m_{2A} - 2m_{1A} + 1)^{|\mathbf{B}|}(1-m_{1A})^{|\mathbf{C}|}\right] E\left[(1-m_{1A})^{|\mathbf{D}|}\right]. \tag{27}$$

However, conditionally to "$N(0,t) = n$", the t_k in $(0,t)$ are independent uniformly distributed random variables. Consequently, the two-dimensional random variable $(|\mathbf{B}|, |\mathbf{C}| \; |N(0,t) = n)$ follows a trinomial law with parameters [5]

$$\left\{\begin{array}{l} b_t = \frac{1}{t}\int_0^t \Pr\left[t'_k - t_k > t+\tau-u\right] du = \frac{1}{t}\int_0^t [1 - G(u+\tau)]\, du \\ c_t = \frac{1}{t}\int_0^t \Pr\left[t - u < t'_k - t_k < t+\tau-u\right] du = \frac{1}{t}\int_0^t [G(u+\tau) - G(u)]\, du \end{array}\right. \tag{28}$$

where $G(u) = \Pr\left[t'_k - t_k < u\right]$. In the same manner, $(|\mathbf{D}| \, |N(0,t) = n)$ is binomial with parameter

$$d = \frac{1}{\tau}\int_t^{t+\tau} \Pr\left[t'_k - t_k > t+\tau-u\right] du = \frac{1}{\tau}\int_0^\tau [1 - G(u)]\, du \tag{29}$$

Expressions (10) are obtained from the multinomial distribution generating function [5]:

$$\left\{\begin{array}{l} E\left[x^{|\mathbf{B}|}y^{|\mathbf{C}|} \, |N(0,t) = n\right] = (xb_t + yc_t + 1 - b_t - c_t)^n \\ E\left[z^{|\mathbf{D}|} \, |N(0,t) = n\right] = (zd + 1 - d)^n \end{array}\right.$$

Consequently:

$$K_I(\tau) = E\left[((m_{2A} - 2m_{1A})\, b_t - m_{1A}c_t + 1)^{N(0,t)}\right] E\left[(1 - m_{1A}d)^{N(t,\tau)}\right]$$

Since $N(0,t)$ and $N(t,\tau)$ are Poisson with parameters λt and $\lambda\tau$, (27) leads to

$$K_I(\tau) = \exp\left[\lambda t\left((m_{2A} - 2m_{1A})\, b_t - m_{1A}c_t\right) - m_{1A}\lambda\tau d\right] \tag{30}$$

The stationary limit is obtained when $t \to \infty$. Finally, (27), (25), (28) lead to:

$$\left\{\begin{array}{ll} K_I(\tau) = \exp\left[\alpha + \beta\int_\tau^\infty (1 - G(u))\, du\right] & \\ m' = E\left[t'_k - t_k\right] & G(u) = \Pr\left[t'_k - t_k < u\right] \\ \alpha = -2\lambda m' m_{1A} & \beta = \lambda m_{2A} \end{array}\right. \tag{31}$$

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using the equality

$$m' = \int_0^\infty (1 - G(u))\, du = \int_0^\infty u dG(u)$$

when this quantity is finite. From a mathematical point of view, the linked characteristic function belongs to the Polya class [8]. Consequently, the mean-square derivative of $I(t)$ cannot exist (despite what was written in [6]). This property is common for processes with discontinuous realizations.

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